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Lie algebra contractions and symmetries of the Toda hierarchy

R Hernández Heredero†‡¶, D Levi†§, M A Rodríguez†‡ and P Winternitz†‖

† Centro Internacional de Ciencias, A.C., Av. Universidad 2001, Col. Chamilpa, Cuernavaca, Morelos, CP 62131, Mexico
‡ Departamento de Física Teórica II, Universidad Complutense, E-28040 Madrid, Spain
§ Dipartimento di Fisica, Università di Roma Tre and INFN-Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Roma, Italy
‖ Centre de Recherches Mathématiques, Université de Montréal, CP 6128, succ Centre-Ville, Montréal PQ H3C 3J7, Canada
¶ Mathematics Department 379 Applied Sciences, University of California at Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, USA

E-mail: rafahh@eucmos.sim.ucm.es, levi@amaldi.fis.uniroma3.it, rodrigue@eucmos.sim.ucm.es and wintern@crm.umontreal.ca

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Abstract. The Lie algebra $L(\Delta)$ of generalized and point symmetries of the equations in the Toda hierarchy is shown to be a semidirect sum of two infinite-dimensional Lie algebras, one perfect, the other Abelian. In the continuous limit the structure of the Lie algebra changes: a contraction occurs with the lattice spacing as the contraction parameter. In particular, for the Toda equation itself, a set of five elements, involving both point symmetries and generalized ones, contracts to the point symmetry algebra of the potential KdV equation.

1. Introduction

The purpose of this paper is to study the infinite-dimensional Lie algebra of point symmetries and higher symmetries of the Toda hierarchy of differential–difference equations [5, 14, 28]. Special emphasis will be on three equations in this hierarchy. The first is the *Toda system*

$$\dot{a}_n = a_n(b_n - b_{n+1}) \qquad \dot{b}_n = a_{n-1} - a_n \tag{1.1}$$

or equivalently the *Toda equation*

$$\ddot{u}_n = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}}. \tag{1.2}$$

The second is the *Volterra equation*

$$\dot{a}_n = a_n(a_{n-1} - a_{n+1}) \tag{1.3}$$

and the third is a *higher Volterra equation*

$$\dot{a}_n = a_n\{a_{n-1}(a_n + a_{n-1} + a_{n-2} - 6) - a_{n+1}(a_{n+2} + a_{n+1} + a_n - 6)\}. \tag{1.4}$$

The point symmetries of the Toda equation were obtained and studied in earlier papers [16, 19]. Point symmetries and higher symmetries of the entire Toda hierarchy were obtained [17] using the Lax pair, i.e. the integrability properties of the hierarchy.

This paper is part of a program, the aim of which is to use Lie theory to study symmetries of discrete equations. The equations are difference equations on a regular lattice. Previous studies have shown [16, 19, 20, 26] that for difference equations the class of point symmetries is somewhat restricted. Many interesting symmetries are obtained only if one considers simultaneous group actions on a finite, or infinite, set of points on the lattice [9, 13, 18] rather than just at one point.

The main result of this paper is that we analyse the infinite-dimensional Lie algebras of point and generalized symmetries of the Toda system. In particular, we show that the continuous limit from a difference equation to a differential one corresponds to a Lie algebra contraction.

Lie algebra contractions were first introduced by Inönü and Wigner [11] to study the relation between relativistic and nonrelativistic theories when the velocity of light c satisfies $c \to \infty$. The Lorentz group then 'contracts' to the Galilei one. A large body of literature now exists on Lie algebra contractions (see [12, 23, 25] and references therein). The contraction parameter in various applications has been the speed of light, Planck's constant, the 'radius' of a space of constant positive or negative curvature, the eccentricity of an ellipse in a nuclear model [4], and others.

In our case the contraction parameter is the lattice spacing h. The infinite-dimensional Lie algebra $L(\Delta)$ of all symmetries of the difference equation contracts to an infinite-dimensional algebra $L(D)$ of symmetries of the differential equation. The crucial new element is that a finite subset $S \subset L(\Delta)$ contracts to a subalgebra L_p of $L(D)$, where L_p is the Lie algebra of point symmetries, while the subset S does not form a Lie algebra. It is not closed under commutation and it contains both point symmetries and generalized symmetries of the discrete equation.

In section 2 we present the symmetries of the Toda hierarchy in a form adapted to our needs. We reproduce some known results [17] to make this paper self-contained. Section 3 is devoted to the structure of the Lie algebra $L(\Delta)$. To obtain the commutation relations we make use of the integrability properties of the hierarchy [8, 22]. We use the spectral transform to relate the nonlinear evolution equations studied to linear equations for the reflection coefficient. Symmetries acting in the solution space of the evolution equation are transformed into symmetries acting in the space of the reflection coefficient. These turn out to be much simpler to deal with. In section 4 we introduce the limiting procedure, taking the Toda equation into the potential Korteweg–de Vries equation and both the Volterra and higher Volterra equation into the Korteweg–de Vries equation itself. The Lie algebras are subjected to the same limiting procedure. Explicitly we present the continuous limits of those symmetries that go into point ones.

2. The Toda hierarchy and its symmetries

The Toda hierarchy is given by the set of nonlinear differential difference equations

$$\begin{pmatrix} \dot{a}_n \\ \dot{b}_n \end{pmatrix} = f_1(\mathcal{L}, t) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} \tag{2.1}$$

associated with the discrete Schrödinger spectral problem

$$\psi(n-1, t; \lambda) + b_n \psi(n, t; \lambda) + a_n \psi(n+1, t; \lambda) = \lambda \psi(n, t; \lambda). \tag{2.2}$$

Above, $f_1(\mathcal{L}, t)$ is an entire function of its first argument, the recursion operator $\mathcal{L}$, given by

$$\mathcal{L} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_n b_{n+1} + a_n(q_n + q_{n+1}) + (b_n - b_{n+1}) s_n \\ b_n q_n + p_n + s_{n-1} - s_n \end{pmatrix} \tag{2.3}$$

where s_n is a solution of the nonhomogeneous first-order equation

$$s_{n+1} = \frac{a_{n+1}}{a_n}(s_n - p_n). \tag{2.4}$$

For any equation of the hierarchy (2.1) we can write down an explicit evolution equation for the function $\psi(n, t; \lambda)$ [2, 3] such that λ does not evolve in time. We impose the following boundary conditions:

$$\lim_{|n|\to\infty} a_n - 1 = \lim_{|n|\to\infty} b_n = \lim_{|n|\to\infty} s_n = 0 \tag{2.5}$$

on the fields a_n, b_n and s_n. We can then associate with equation (2.2) a spectrum defined in the complex plane of the variable $z(\lambda = z + z^{-1})$:

$$\{R(z,t),\ z \in \boldsymbol{C}_1; z_j, c_j(t), |z_j| < 1, j = 1, 2, \dots, N\} \tag{2.6}$$

where $R(z, t)$ is the reflection coefficient, $\boldsymbol{C}_1$ is the unit circle in the complex z plane, z_j are isolated points inside the unit disk and c_j are some complex functions of t related to the residues of $R(z, t)$ at the poles z_j. When a_n, b_n and s_n satisfy the boundary conditions (2.5), the spectral data defines the potentials in a unique way.

Thus, there is a one-to-one correspondence between the evolution of the potentials (a_n, b_n) of the discrete Schrödinger spectral problem (2.2), given by equation (2.1) and that of the reflection coefficient $R(\lambda, t)$, given by

$$\frac{\mathrm{d}R(z,t)}{\mathrm{d}t} = \mu f_1(\lambda, t) R(z,t) \qquad \mu = z^{-1} - z. \tag{2.7}$$

In equations (2.7) and below, $\frac{\mathrm{d}}{\mathrm{d}y}$ denotes the total derivative with respect to y.

The Toda system is obtained from equation (2.1) by choosing $f_1(\lambda, t) = 1$ and thus the evolution equation of the reflection coefficient is given by

$$\frac{\mathrm{d}R(z,t)}{\mathrm{d}t} = \mu R(z,t). \tag{2.8}$$

The Toda equation (1.2) is obtained from the Toda system by setting

$$b_n = \dot{u}_n \qquad a_n = \mathrm{e}^{u_n - u_{n+1}}. \tag{2.9}$$

The Volterra (1.3) and higher Volterra (1.4) equations are obtained from equation (2.1) by setting $b_n = 0$. The Toda hierarchy (2.1) then reduces to what from now on we will call the Volterra hierarchy:

$$\dot{a}_n = g_1(\tilde{\mathcal{L}}, t)\{a_n(a_{n-1} - a_{n+1})\} \tag{2.10}$$

where we have

$$\tilde{\mathcal{L}} p_n = a_n(p_n + p_{n+1} + s_{n-1} - s_{n+1}) \tag{2.11}$$

with s_n given by equation (2.4). In correspondence with any equation of the class (2.10) we can define the evolution of the reflection coefficient $R(z, t)$ of the associated Schrödinger spectral problem (2.2), given by

$$\frac{\mathrm{d}R(z,t)}{\mathrm{d}t} = \mu\lambda g_1(\lambda^2, t) R(z,t). \tag{2.12}$$

The Volterra equation (1.3) is obtained for $g_1(\lambda^2, t) = 1$ while the higher Volterra (1.4) is obtained for $g_1(\lambda^2, t) = \lambda^2 - 4$.

The symmetries for any equation of the Toda (2.1) and Volterra (2.10) hierarchies are provided by all flows commuting with the equations themselves. An infinite number of such symmetries is provided by the equations

$$\begin{pmatrix} a_{n,\epsilon_k} \\ b_{n,\epsilon_k} \end{pmatrix} = \mathcal{L}^k \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} \tag{2.13}$$

in the case of the Toda system equations (1.1) and

$$a_{n,\epsilon_k} = (\tilde{\mathcal{L}})^k \{a_n(a_{n-1} - a_{n+1})\} \tag{2.14}$$

for the Volterra hierarchy. Here k is any positive integer and ϵ_k is a group parameter. From the point of view of the spectral problem (2.2) the equations above correspond to isospectral deformations, i.e., we have $\lambda_{\epsilon_k} = 0$. For any ϵ_k, the solution of the Cauchy problem for equation (2.13) provides a solution of one of the Toda hierarchy equations $(a_n(\epsilon_k), b_n(\epsilon_k))$ in terms of the initial condition $(a_n(\epsilon_k = 0), b_n(\epsilon_k = 0))$ (and similarly equation (2.14) for the Volterra equation). The group transformation corresponding to the group parameter ϵ_k can usually be written explicitly only for a few values of k. In all other cases one can just use the symmetries (2.13) to do, for example, symmetry reduction and to reduce the equation under consideration to an ordinary differential equation, or possibly a functional one. The proof of the validity of equation (2.13) is easily given by taking into account the one-to-one correspondence between the equation and the spectrum (2.6), under the asymptotic conditions (2.5). In fact, in such a case we can biunivocally associate with both the equation (2.1) and the symmetries (2.13) an evolution of the reflection coefficient. In the case of the symmetries (2.13), we have

$$\frac{\mathrm{d}R}{\mathrm{d}\epsilon_k} = \mu\lambda^k R. \tag{2.15}$$

It is easy to prove that the flows (2.7) and (2.15) commute and hence the same must be true for the corresponding equations for the fields ((2.1) and (2.13)). This has also been verified directly for low values of k. The results for the Volterra hierarchy are quite analogous, but the evolution of the reflection coefficient for equation (2.14) is given by

$$\frac{\mathrm{d}R}{\mathrm{d}\epsilon_k} = \mu\lambda^{2k+1} R. \tag{2.16}$$

We can extend the class of symmetries by considering nonisospectral deformations of the spectral problem (2.2) [14]. Thus for the Toda hierarchy we obtain

$$\begin{pmatrix} a_{n,\epsilon_k} \\ b_{n,\epsilon_k} \end{pmatrix} = f_2(\mathcal{L}, t) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} + \mathcal{L}^k \begin{pmatrix} a_n[(2n+3)b_{n+1} - (2n-1)b_n] \\ b_n^2 - 4 + 2[(n+1)a_n - (n-1)a_{n-1}] \end{pmatrix} \tag{2.17a}$$

where the function $f_2(\mathcal{L}, t)$ is obtained as a solution of the differential equation:

$$f_2(\mathcal{L}, t)_t = \mathcal{L}^k \left[(\mathcal{L}^2 - 4) \frac{\partial f_1(\mathcal{L}, t)}{\partial \mathcal{L}} + \mathcal{L} f_1(\mathcal{L}, t) \right]. \tag{2.17b}$$

Thus $f_2(\mathcal{L}, t)$ is expressed in terms of the function $f_1(\mathcal{L}, t)$ where f_1 defines the equation under consideration. Similarly in the case of the Volterra hierarchy, we have

$$a_{n,\epsilon_k} = g_2(\tilde{\mathcal{L}}, t) \left[a_n(a_{n-1} - a_{n+1}) \right] + \tilde{\mathcal{L}}^k [a_n(a_n - (n-1)a_{n-1} + (n+2)a_{n+1} - 4)] \tag{2.18a}$$

with

$$g_2(\tilde{\mathcal{L}}, t)_t = \tilde{\mathcal{L}}^k \left[\tilde{\mathcal{L}}(\tilde{\mathcal{L}} - 4) \frac{\mathrm{d}g_1(\tilde{\mathcal{L}}, t)}{\mathrm{d}\tilde{\mathcal{L}}} + (\tilde{\mathcal{L}} - 2) g_1(\tilde{\mathcal{L}}, t) \right]. \tag{2.18b}$$

In correspondence to equation (2.17a) we have the evolution of the reflection coefficient (2.6), given by

$$\frac{\mathrm{d}R}{\mathrm{d}\epsilon_k} = \mu f_2(\lambda, t) R \qquad \lambda_{\epsilon_k} = \mu^2 \lambda^k \tag{2.19}$$

while for equation (2.18a) we have

$$\frac{\mathrm{d}R}{\mathrm{d}\epsilon_k} = \mu\lambda g_2(\lambda^2, t) R \qquad \lambda_{\epsilon_k} = \tfrac{1}{2}\mu^2 \lambda^{2k+1}. \tag{2.20}$$

As in the case of isospectral symmetries (2.13) and (2.14), we must prove that the nonisospectral flows (2.17*a*) and (2.18*a*) commute with the corresponding evolution equations (2.1) and (2.10). This is reduced to an easier task, namely showing that the flows (2.19) and (2.20) in the space of the reflection coefficients commute with the evolution equations (2.7) and (2.12), respectively.

In addition to the above two hierarchies of symmetries (2.13) and (2.17*a*) for the Toda hierarchy, we can construct two further symmetries, which, however, do not satisfy the asymptotic boundary conditions (2.5). They are

$$\begin{pmatrix} a_{n,\epsilon} \\ b_{n,\epsilon} \end{pmatrix} = f_3(\mathcal{L}, t) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.21}$$

$$\begin{pmatrix} a_{n,\epsilon} \\ b_{n,\epsilon} \end{pmatrix} = f_4(\mathcal{L}, t) \begin{pmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{pmatrix} + \begin{pmatrix} 2a_n \\ b_n \end{pmatrix} \tag{2.22}$$

where the functions $f_3(\mathcal{L}, t)$ and $f_4(\mathcal{L}, t)$ are to be determined directly for each equation of the hierarchy. In the case of the Volterra hierarchy we have only one exceptional symmetry, given by

$$a_{n,\epsilon} = g_3(\tilde{\mathcal{L}}, t)[a_n(a_{n-1} - a_{n+1})] + a_n \tag{2.23}$$

where, as for the case of equations (2.21) and (2.22), the function $g_3(\tilde{\mathcal{L}}, t)$ is to be determined directly for each equation of the hierarchy. As these exceptional symmetries do not satisfy the asymptotic boundary conditions (2.5), we cannot write a corresponding evolution equation for the reflection coefficient (2.6).

Let us now write down the lowest-order symmetries for the specific equations we are treating in detail, i.e. the Toda equation (1.1), the Volterra equation (1.3) and the higher Volterra equation (1.4). In the case of the Toda equation (1.2) the symmetries are obtained from those of the Toda system by using transformation (2.9). The symmetries of the Toda lattice and the Toda system corresponding to the isospectral and nonisospectral flows will have the same evolution of the reflection coefficient. Transformation (2.9) involves an integration (to obtain u_n). The integration constant must be chosen so as to satisfy the following boundary conditions:

$$\lim_{|n|\to\infty} u_n = 0. \tag{2.24}$$

In the case of the exceptional symmetries such an integration will provide an additional symmetry and, moreover, the resulting symmetries will be defined up to integration constants.

Taking $k = 0$, 1, and 2 in equation (2.13) we obtain the first three isospectral symmetries for the Toda system, namely:

$$a_{n,\epsilon_0} = a_n(b_n - b_{n+1}) \qquad b_{n,\epsilon_0} = a_{n-1} - a_n \tag{2.25}$$

$$\begin{aligned} a_{n,\epsilon_1} &= a_n(b_n^2 - b_{n+1}^2 + a_{n-1} - a_{n+1}) \\ b_{n,\epsilon_1} &= a_{n-1}(b_n + b_{n-1}) - a_n(b_{n+1} + b_n) \end{aligned} \tag{2.26}$$

$$\begin{aligned} a_{n,\epsilon_2} &= a_n(b_n^3 - b_{n+1}^3 + a_n b_n - 2a_{n+1}b_{n+1} + a_{n-1}b_{n-1} + 2a_{n-1}b_n \\ &\qquad - a_{n+1}b_{n+2} - a_n b_{n+1} - 2b_n + 2b_{n+1}) \\ b_{n,\epsilon_2} &= a_{n-1}(b_n^2 + b_{n-1}^2 + b_n b_{n-1} + a_{n-1} + a_{n-2} - 2) \\ &\qquad - a_n(b_n^2 + b_{n+1}^2 + b_n b_{n+1} + a_{n+1} + a_n - 2). \end{aligned} \tag{2.27}$$

The lowest nonisospectral symmetry is obtained from equation (2.17*a*), taking $k = 0$. It is

$$\begin{aligned} a_{n,\nu} &= a_n\{t(b_n^2 - b_{n+1}^2 + a_{n-1} - a_{n+1}) + (2n+3)b_{n+1} - (2n-1)b_n\} \\ b_{n,\nu} &= t\{a_{n-1}(b_n + b_{n-1}) - a_n(b_{n+1} + b_n)\} + b_n^2 - 4 + 2[(n+1)a_n - (n-1)a_{n-1}]. \end{aligned} \tag{2.28}$$

The exceptional symmetries (2.21) and (2.22) are

$$a_{n,\mu_0} = 0 \qquad b_{n,\mu_0} = 1 \tag{2.29}$$

$$a_{n,\mu_1} = 2a_n + t\dot{a}_n \qquad b_{n,\mu_1} = b_n + t\dot{b}_n. \tag{2.30}$$

Definition (2.9) allows us to obtain the corresponding symmetries for the Toda equation:

$$u_{n,\epsilon_0} = \dot{u}_n \tag{2.31}$$

$$u_{n,\epsilon_1} = \dot{u}_n^2 + \mathrm{e}^{u_{n-1}-u_n} + \mathrm{e}^{u_n-u_{n+1}} - 2 \tag{2.32}$$

$$u_{n,\epsilon_2} = \dot{u}_n^3 - 2\dot{u}_n + \mathrm{e}^{u_{n-1}-u_n}(\dot{u}_{n-1} + 2\dot{u}_n) + \mathrm{e}^{u_n-u_{n+1}}(\dot{u}_{n+1} + 2\dot{u}_n) \tag{2.33}$$

$$u_{n,\nu} = t\{\dot{u}_n^2 + \mathrm{e}^{u_{n-1}-u_n} + \mathrm{e}^{u_n-u_{n+1}} - 2\} - (2n-1)\dot{u}_n + w_n(t) \tag{2.34}$$

where $w_n(t)$ is defined by the following compatible system of equations:

$$w_{n+1}(t) - w_n(t) = -2\dot{u}_{n+1} \qquad \dot{w}_n(t) = 2(\mathrm{e}^{u_n-u_{n+1}} - 1). \tag{2.35}$$

Under the assumption (2.24) we could integrate equations (2.35) and obtain a formal solution. That is, we can write $w_n(t)$ in the form of an infinite sum:

$$w_n(t) = 2\sum_{j=n+1}^{\infty} \dot{u}_j + \alpha \tag{2.36}$$

where α is an arbitrary integration constant which can be interpreted as an additional symmetry. However, the additional freedom provided by the use of equations (2.35) instead of its solution, given by equation (2.36), will be put to good use later in the calculation of the commutators and will allow us to take continuous limits. The exceptional symmetries are

$$u_{n,\mu_1} = t\dot{u}_n - 2n \tag{2.37}$$

$$u_{n,\mu_0} = t \tag{2.38}$$

and the additional one, due to the integration, is

$$u_{n,\mu_{-1}} = 1. \tag{2.39}$$

In the case of the Volterra equation, we have

$$a_{n,\epsilon_0} = a_n(a_{n-1} - a_{n+1}) \tag{2.40}$$

$$a_{n,\epsilon_1} = a_n\{a_{n-1}(a_{n-2} + a_{n-1} + a_n - 2) - a_{n+1}(a_{n+2} + a_{n+1} + a_n - 2)\} \tag{2.41}$$

$$\begin{aligned} a_{n,\epsilon_2} = a_n\{&a_{n-1}[(a_n + a_{n-1})(a_{n-2} + a_{n-1} + a_n - 2) \\ &+a_{n-2}(a_{n-3} + a_{n-2} + a_{n-1} - 2) - 2] \\ &-a_{n+1}[(a_{n+1} + a_n)(a_{n+2} + a_{n+1} + a_n - 2) \\ &+a_{n+2}(a_{n+3} + a_{n+2} + a_{n+1} - 2) - 2]\} \end{aligned} \tag{2.42}$$

$$\begin{aligned} a_{n,\nu} = a_n\{&t[a_{n-1}(a_{n-2} + a_{n-1} + a_n - 4) - a_{n+1}(a_{n+2} + a_{n+1} + a_n - 4)] \\ &+a_n - (n-1)a_{n-1} + (n+2)a_{n+1} - 4\} \end{aligned} \tag{2.43}$$

and the exceptional one

$$a_{n,\mu} = a_n + t\dot{a}_n. \tag{2.44}$$

In the case of the higher Volterra equation the isospectral symmetries are the same as those ((2.40)–(2.42)) of the Volterra equations. The nonisospectral and exceptional ones are different:

$$\begin{aligned} a_{n,\nu} = a_n\{&a_n - (n-1)a_{n-1} + (n+2)a_{n+1} - 4 + 2t[a_{n-1}[a_{n-2}(a_{n-3} + a_{n-2} + a_{n-1}) \\ &+(a_{n-2} + a_{n-1} + a_n)(a_{n-1} + a_n - 7) + 12] - a_{n+1}[a_{n+2}(a_{n+3} + a_{n+2} \\ &+a_{n+1}) + (a_{n+2} + a_{n+1} + a_n)(a_{n+1} + a_n - 7) + 12]]\} \end{aligned} \tag{2.45}$$

$$a_{n,\mu} = a_n\{1 + 2t[a_{n-1}(a_{n-2} + a_{n-1} + a_n - 3) - a_{n+1}(a_{n+2} + a_{n+1} + a_n - 3)]\}. \tag{2.46}$$

3. Commutation relations

To define the structure of the symmetry algebra for the Toda and Volterra hierarchies we need to compute the commutation relations between the symmetries, i.e. the flows commuting with the equations of the hierarchy. Apart from the exceptional cases, the proof that the symmetry flows commute with the equations has been carried out using the one-to-one correspondence between the integrable equations and the evolution equations for the reflection coefficients. Without using this correspondence it would be extremely difficult to prove the existence of an infinite denumerable number of symmetries.

In this section we extend the approach using the reflection coefficients to calculate the commutation relations between the symmetries and thus to analyse the structure of the obtained infinite-dimensional Lie algebra.

The first result is that the isospectral symmetry generators, provided by equations (2.13) and (2.14) for the Toda system and Volterra equation, respectively, commute amongst each other. If we define

$$\mathcal{L}^k = \begin{pmatrix} \mathcal{L}_{11}^{(k)} & \mathcal{L}_{12}^{(k)} \\ \mathcal{L}_{21}^{(k)} & \mathcal{L}_{22}^{(k)} \end{pmatrix} \tag{3.1}$$

we can write the generators for the isospectral symmetries

$$\hat{X}_k^T = \{\mathcal{L}_{11}^{(k)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{12}^{(k)}(a_{n-1} - a_n)\}\partial_{a_n} + \{\mathcal{L}_{21}^{(k)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{22}^{(k)}(a_{n-1} - a_n)\}\partial_{b_n} \tag{3.2}$$

and

$$\hat{X}_k^V = \tilde{\mathcal{L}}^k[a_n(a_{n-1} - a_{n+1})]\partial_{a_n} \tag{3.3}$$

for the Toda and Volterra hierarchies, respectively. The fact, proven in section 2, that equations (2.13) and (2.14) provide symmetries for a generic equation of the Toda and Volterra hierarchies, implies

$$[\hat{X}_k, \hat{X}_m] = 0. \tag{3.4}$$

Indeed, the relation

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\epsilon_k\,\mathrm{d}\epsilon_m} = \frac{\mathrm{d}^2 R}{\mathrm{d}\epsilon_m\,\mathrm{d}\epsilon_k} \tag{3.5}$$

follows directly from equation (2.15). A natural way of representing the result given by equation (3.5) is to introduce symmetry generators in the space of the reflection coefficients. These generators are written as

$$\hat{\mathcal{X}}_k^T = \mu\lambda^k R\partial_R \qquad \text{and} \qquad \hat{\mathcal{X}}_k^V = \mu\lambda^{2k+1} R\partial_R \tag{3.6}$$

for the Toda and Volterra hierarchies, respectively. In agreement with Lie theory, whenever R is an analytic function of ϵ, the corresponding flows are given by solving the equations

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}\epsilon_k} = \mu\lambda^k\tilde{R} \qquad \frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\epsilon_k} = 0 \qquad \tilde{R}(\epsilon_k = 0) = R \qquad \tilde{\lambda}(\epsilon_k = 0) = \lambda \tag{3.7}$$

for the Toda hierarchy, and

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}\epsilon_k} = \mu\lambda^{2k+1}\tilde{R} \qquad \frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\epsilon_k} = 0 \qquad \tilde{R}(\epsilon_k = 0) = R \qquad \tilde{\lambda}(\epsilon_k = 0) = \lambda \tag{3.8}$$

for the Volterra one. These equations coincide with equation (2.15). In terms of the vector fields $\hat{\mathcal{X}}_k$, equation (3.5) is written as

$$[\hat{\mathcal{X}}_k, \hat{\mathcal{X}}_m] = [\mu\lambda^k R\partial_R, \mu\lambda^m R\partial_R] = 0. \tag{3.9}$$

So far, the use of the vector fields in the reflection coefficient space has just reexpressed a known result, namely equation (3.5) is rewritten as equation (3.9). We now extend the use of vector fields in the reflection coefficient space to the case of the nonisospectral symmetries (2.17*a*) and (2.18*a*). Using the definition (3.1) we can introduce the generators of the nonisospectral symmetries for the Toda and Volterra hierarchies. We restrict ourselves, for the sake of the simplicity of exposition, to equations of the Toda and Volterra hierarchies for which there is no explicit dependence on time. That is, we put

$$f_1(\lambda, t) = \lambda^N \qquad g_1(\lambda^2, t) = \lambda^{2N} \qquad N \in \mathbf{Z}^+. \tag{3.10}$$

This makes it possible to integrate equations (2.17*b*) and (2.18*b*) to obtain

$$f_2 = \lambda^{k+N-1}[(1+N)\lambda^2 - 4N]t \qquad g_2 = \lambda^{2N+2k}[(1+N)\lambda^2 - 2(2N+1)]t. \tag{3.11}$$

The symmetry vector fields for the Toda and Volterra hierarchies are now

$$\begin{aligned}\hat{Y}_k^T = \{ & t(1+N)[\mathcal{L}_{11}^{(k+N+1)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{12}^{(k+N+1)}(a_{n-1} - a_n)] \\ & -4Nt[\mathcal{L}_{11}^{(k+N-1)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{12}^{(k+N-1)}(a_{n-1} - a_n)] \\ & +\mathcal{L}_{11}^{(k)}[a_n((2n+3)b_{n+1} - (2n-1)b_n)] \\ & +\mathcal{L}_{12}^{(k)}[b_n^2 - 4 + 2(n+1)a_n - 2(n-1)a_{n-1}]\}\partial_{a_n} \\ & +\{t(1+N)[\mathcal{L}_{21}^{(k+N+1)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{22}^{(k+N+1)}(a_{n-1} - a_n)] \\ & -4Nt[\mathcal{L}_{21}^{(k+N-1)}[a_n(b_n - b_{n+1})] + \mathcal{L}_{22}^{(k+N-1)}(a_{n-1} - a_n)] \\ & +\mathcal{L}_{21}^{(k)}[a_n((2n+3)b_{n+1} - (2n-1)b_n)] \\ & +\mathcal{L}_{22}^{(k)}[b_n^2 - 4 + 2(n+1)a_n - 2(n-1)a_{n-1}]\}\partial_{b_n}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\hat{Y}_k^V = \{ & t\tilde{\mathcal{L}}^{k+N}[(1+N)\tilde{\mathcal{L}} - 2(1+2N)][a_n(a_{n-1} - a_{n+1})] \\ & +\tilde{\mathcal{L}}^k[a_n(a_n - (n-1)a_{n-1} + (n+2)a_{n+1} - 4)]\}\partial_{a_n}\end{aligned} \tag{3.13}$$

respectively. Taking into account equations (2.19), (2.20) and (3.11) we can define the symmetry generators (3.12) and (3.13) in the reflection coefficient space, i.e.

$$\hat{\mathcal{Y}}_k^T = \mu\lambda^{k+N-1}t[(1+N)\lambda^2 - 4N]R\partial_R + \mu^2\lambda^k\partial_\lambda \tag{3.14}$$

$$\hat{\mathcal{Y}}_k^V = \mu\lambda^{2k+2N+1}t[(1+N)\lambda^2 - 4N - 2]R\partial_R + \tfrac{1}{2}\mu^2\lambda^{2k+1}\partial_\lambda. \tag{3.15}$$

Commuting $\hat{\mathcal{Y}}_k$ with $\hat{\mathcal{Y}}_m$ we have

$$[\hat{\mathcal{Y}}_k^T, \hat{\mathcal{Y}}_m^T] = (m-k)[\hat{\mathcal{Y}}_{k+m+1}^T - 4\hat{\mathcal{Y}}_{k+m-1}^T] \tag{3.16}$$

$$[\hat{\mathcal{Y}}_k^V, \hat{\mathcal{Y}}_m^V] = (m-k)[\hat{\mathcal{Y}}_{k+m+1}^V - 4\hat{\mathcal{Y}}_{k+m}^V]. \tag{3.17}$$

From the isomorphism between the spectral space and the space of the solutions, we conclude that the vector fields representing the symmetries of the studied evolution equations satisfy the same commutation relations. Hence we have

$$[\hat{Y}_k^T, \hat{Y}_m^T] = (m-k)[\hat{Y}_{k+m+1}^T - 4\hat{Y}_{k+m-1}^T] \tag{3.18}$$

$$[\hat{Y}_k^V, \hat{Y}_m^V] = (m-k)[\hat{Y}_{k+m+1}^V - 4\hat{Y}_{k+m}^V]. \tag{3.19}$$

In a similar manner we can work out the commutation relations between the $\hat{Y}_k$ and $\hat{X}_m$ symmetry generators. We get

$$[\hat{\mathcal{X}}_k^T, \hat{\mathcal{Y}}_m^T] = -(1+k)\hat{\mathcal{X}}_{k+m+1}^T + 4k\hat{\mathcal{X}}_{k+m-1}^T \tag{3.20}$$

$$[\hat{\mathcal{X}}_k^V, \hat{\mathcal{Y}}_m^V] = -(1+k)\hat{\mathcal{X}}_{k+m+1}^V + 2(2k+1)\hat{\mathcal{X}}_{k+m}^V \tag{3.21}$$

and consequently

$$[\hat{X}^T_k, \hat{Y}^T_m] = -(1+k)\hat{X}^T_{k+m+1} + 4k\hat{X}^T_{k+m-1} \tag{3.22}$$

$$[\hat{X}^V_k, \hat{Y}^V_m] = -(1+k)\hat{X}^V_{k+m+1} + 2(2k+1)\hat{X}^V_{k+m}. \tag{3.23}$$

Let us now consider the commutation relations involving the exceptional symmetries (2.21)–(2.23). As mentioned in section 2, these symmetries do not satisfy the asymptotic conditions (2.5). Hence we cannot write them in the space of the reflection coefficient and we cannot write them down simultaneously for all equations in the hierarchy. Consequently we must consider each case separately. In the case of the Toda system ($N = 0$) we have two exceptional symmetries $\hat{Z}_0$, $\hat{Z}_1$. Using equations (2.29) and (2.30) we write them as

$$\hat{Z}^T_0 = \partial_{b_n} \tag{3.24}$$

$$\hat{Z}^T_1 = [2a_n + t\dot{a}_n]\partial_{a_n} + [b_n + t\dot{b}_n]\partial_{b_n}. \tag{3.25}$$

We can then calculate explicitly the commutation relations involving $\hat{Z}^T_0$ and $\hat{Z}^T_1$, $\hat{X}^T_0$, $\hat{X}^T_1$ and $\hat{Y}^T_0$. The nonzero commutation relations are

$$\begin{array}{lll} [\hat{X}^T_0, \hat{Z}^T_1] = -\hat{X}^T_0 & [\hat{Z}^T_0, \hat{Z}^T_1] = \hat{Z}^T_0 & \\ [\hat{Y}^T_0, \hat{Z}^T_0] = -2\hat{Z}^T_1 & [\hat{Y}^T_0, \hat{Z}^T_1] = -\hat{Y}^T_0 - 8\hat{Z}^T_0 & [\hat{X}^T_1, \hat{Z}^T_0] = -2\hat{X}^T_0 \\ [\hat{X}^T_1, \hat{Z}^T_1] = -2\hat{X}^T_1 & [\hat{X}^T_0, \hat{Y}^T_0] = -\hat{X}^T_1 & [\hat{X}^T_1, \hat{Y}^T_0] = -2\hat{X}^T_2 + 4\hat{X}^T_0. \end{array} \tag{3.26}$$

In the case of the Toda equation we have (see equations (2.37) and (2.38))

$$\hat{Z}^{TE}_0 = t\partial_{u_n} \tag{3.27}$$

$$\hat{Z}^{TE}_1 = [t\dot{u}_n - 2n]\partial_{u_n} \tag{3.28}$$

and

$$\hat{Z}^{TE}_{-1} = \partial_{u_n} \tag{3.29}$$

in correspondence with equation (2.39). The symmetries $\hat{X}^{TE}_0$, $\hat{X}^{TE}_1$ and $\hat{Y}^{TE}_0$, according to equations (2.31), (2.32) and (2.34), are given by

$$\begin{array}{l} \hat{X}^{TE}_0 = \dot{u}_n\partial_{u_n} \qquad \hat{X}^{TE}_1 = [\dot{u}^2_n + \mathrm{e}^{u_{n-1}-u_n} + \mathrm{e}^{u_n-u_{n+1}} - 2]\partial_{u_n} \\ \hat{Y}^{TE}_0 = \{t[u^2_{n,t} + \mathrm{e}^{u_{n-1}-u_n} + \mathrm{e}^{u_n-u_{n+1}} - 2] - (2n-1)u_{n,t} + w_n(t)\}\partial_{u_n} \\ w_{n+1}(t) - w_n(t) = -2\dot{u}_{n+1} \qquad \dot{w}_n(t) = 2(\mathrm{e}^{u_n-u_{n+1}} - 1). \end{array} \tag{3.30}$$

The nonzero commutation relations are

$$\begin{array}{ll} [\hat{X}^{TE}_0, \hat{Z}^{TE}_0] = -\hat{Z}^{TE}_{-1} & [\hat{X}^{TE}_0, \hat{Z}^{TE}_1] = -\hat{X}^{TE}_0 \\ [\hat{X}^{TE}_0, \hat{Y}^{TE}_0] = -\hat{X}^{TE}_1 + \omega\hat{Z}^{TE}_{-1} & \\ [\hat{X}^{TE}_1, \hat{Z}^{TE}_0] = -2\hat{X}^{TE}_0 & [\hat{X}^{TE}_1, \hat{Z}^{TE}_1] = -2\hat{X}^{TE}_1 - 4\hat{Z}^{TE}_{-1} \\ [\hat{X}^{TE}_1, \hat{Y}^{TE}_0] = -2\hat{X}^{TE}_2 + 4\hat{X}^{TE}_0 + \sigma\hat{Z}^{TE}_{-1} & \\ [\hat{Y}^{TE}_0, \hat{Z}^{TE}_{-1}] = \beta\hat{Z}^{TE}_{-1} & [\hat{Y}^{TE}_0, \hat{Z}^{TE}_0] = -2\hat{Z}^{TE}_1 + \gamma\hat{Z}^{TE}_{-1} \\ [\hat{Y}^{TE}_0, \hat{Z}^{TE}_1] = -\hat{Y}^{TE}_0 - 8\hat{Z}^{TE}_0 + \delta\hat{Z}^{TE}_{-1} & [\hat{Z}^{TE}_0, \hat{Z}^{TE}_1] = \hat{Z}^{TE}_0 \end{array} \tag{3.31}$$

where $(\beta, \gamma, \delta, \omega, \sigma)$ are integration constants. Notice that the presence of these integration constants indicates that the symmetry algebra of the Toda equation is not completely specified. The constants appear whenever the symmetry $\hat{Y}^{TE}_0$ is involved. The ambiguity is related to the ambiguity in the definition of $\hat{Y}^{TE}_0$ itself, i.e. in the solution of equations (3.30) for $w_n(t)$. To remove this ambiguity, supplementary conditions must be involved. In section 4 we shall see that all the constants are specified by requiring that one obtains the correct continuous limit.

In the case of the equations of the Volterra hierarchy we have only one exceptional symmetry. For the Volterra equation it is (see equation (2.44))

$$\hat{Z}^V = [a_n + t\dot{a}_n]\partial_{a_n}. \tag{3.32}$$

From equations (2.40) and (2.43) we obtain the lowest symmetries in the $\hat{X}$ and $\hat{Y}$ series. Commuting explicitly, we obtain

$$\begin{array}{lll} [\hat{Z}^V, \hat{X}_0^V] = \hat{X}_0^V & [\hat{Z}^V, \hat{Y}_0^V] = \hat{Y}_0^V + 4\hat{Z}^V & [\hat{Z}^V, \hat{X}_1^V] = 2(\hat{X}_0^V + \hat{X}_1^V) \\ [\hat{Y}_0^V, \hat{X}_0^V] = \hat{X}_1^V - 2\hat{X}_0^V & [\hat{Y}_0^V, \hat{X}_1^V] = 2\hat{X}_2^V - 6\hat{X}_1^V. & \end{array} \tag{3.33}$$

For the higher Volterra we have

$$\hat{Z}^{HV} = a_n\{1 + 2t[a_{n-1}(a_n + a_{n-1} + a_{n-2} - 3) - a_{n+1}(a_n + a_{n+1} + a_{n+2} - 3)]\}\partial_{a_n} \tag{3.34}$$

and we obtain

$$\begin{array}{ll} [\hat{Z}^{HV}, \hat{X}_0^{HV}] = \hat{X}_0^{HV} & [\hat{Z}^{HV}, \hat{Y}_0^{HV}] = \hat{Y}_0^{HV} + 4\hat{Z}^{HV} \\ [\hat{Y}_0^{HV}, \hat{X}_0^{HV}] = \hat{X}_1^{HV} - 2\hat{X}_0^{HV} & [\hat{Y}_0^{HV}, \hat{X}_1^{HV}] = 2\hat{X}_2^{HV} - 6\hat{X}_1^{HV} \\ [\hat{Z}^{HV}, \hat{X}_1^{HV}] = 2(\hat{X}_0^{HV} + \hat{X}_1^{HV}). & \end{array} \tag{3.35}$$

The generators $\hat{Y}_k$ and $\hat{\mathcal{Y}}_k$ (see equations (3.12)–(3.15)) depend on the number N, which enumerates equations in the hierarchy. Interestingly, the commutation relations involving the generators $\hat{X}$ and $\hat{Y}$ are the same for all N (see equations (3.4), (3.9) and (3.16)–(3.23)).

The commutation relations obtained above determine the structure of the infinite-dimensional Lie symmetry algebras. For the Toda system the first symmetry generators are given in equations (3.2), (3.12), (3.24) and (3.25) and the corresponding commutation relations are given by equations (3.18), (3.22) and (3.26). As one can see, the symmetry operators $\hat{Y}_k$ and $\hat{Z}_k$ are linear in t and its coefficient is an isospectral symmetry $\hat{X}_k$. Consequently, as the operators $\hat{X}_k$ commute among themselves, the commutator of $\hat{X}_m$ with any of the $\hat{Y}_k$ or $\hat{Z}_k$ symmetries will not have any explicit time dependence and thus can be written in terms of $\hat{X}_n$ only. The structure of the Lie algebra for the Toda system can be written as

$$L = L_0 \ni\!\!\!\!\supset L_1 \qquad L_0 = \{\hat{h}, \hat{e}, \hat{f}, \hat{Y}_1^T, \hat{Y}_2^T, \ldots\} \qquad L_1 = \{\hat{X}_0^T, \hat{X}_1^T, \ldots\} \tag{3.36}$$

where $\{\hat{h} = \hat{Z}_1^T, \hat{e} = \hat{Z}_0^T, \hat{f} = \hat{Y}_0^T + 4\hat{Z}_0^T\}$ denotes a $sl(2, \boldsymbol{R})$ subalgebra with $[\hat{h}, \hat{e}] = \hat{e}$, $[\hat{h}, \hat{f}] = -\hat{f}$, $[\hat{e}, \hat{f}] = 2\hat{h}$. The algebra L_0 is perfect, i.e., we have $[L_0, L_0] = L_0$. Let us point out that $\hat{Z}_0^T$, $\hat{Z}_1^T$ and $\hat{X}_0^T$ are point symmetries, all others are generalized symmetries.

For the Toda equation the point transformations are $\hat{X}_0^{TE}$, $\hat{Z}_0^{TE}$ and $\hat{Z}_1^{TE}$, as for the Toda system, plus the additional $\hat{Z}_{-1}^{TE}$. Taking into account equations (3.27)–(3.31), the structure of the Lie algebra is as in equations (3.36) with $L_0 = \{\hat{Z}_0^{TE}, \hat{Z}_1^{TE}, \hat{Y}_0^{TE}, \hat{Y}_1^{TE}, \hat{Y}_2^{TE}, \ldots\}$ and $L_1 = \{\hat{Z}_{-1}^{TE}, \hat{X}_0^{TE}, \hat{X}_1^{TE}, \hat{X}_2^{TE}, \ldots\}$. The algebra L has a finite-dimensional subalgebra $\{\hat{Z}_0^{TE}, \hat{Z}_1^{TE}, \hat{Y}_0^{TE}, \hat{Z}_{-1}^{TE}\}$ isomorphic to $gl(2, \boldsymbol{R}) \sim \{\hat{Z}_0^{TE}, \hat{Z}_1^{TE}, \hat{Y}_0^{TE}\} \oplus \{\hat{Z}_{-1}^{TE}\}$. The element $\hat{Z}_{-1}^{TE}$ is in the centre of L and L_0 is a perfect Lie algebra.

For the Volterra equation $\hat{X}_0^V$ and $\hat{Z}^V$ are point symmetries. All the other symmetries are higher ones. Taking into account equations (3.3), (3.13), (3.32) and (3.33), the structure of the Lie algebra is again $L = L_0 \ni\!\!\!\!\supset L_1$ with $L_0 = \{\hat{Z}^V, \hat{Y}_0^V, \hat{Y}_1^V, \hat{Y}_2^V, \ldots\}$ and $L_1 = \{\hat{X}_0^V, \hat{X}_1^V, \ldots\}$.

For the higher Volterra equation $\hat{X}_i^{HV}$ are as in equation (3.3); $\hat{Y}_0^{HV}$ and $\hat{Z}^{HV}$ are somewhat different (see equation (3.13) with $N = 1$ and $k = 0$ and (3.34)). The commutation relations (3.35), and hence the structure of the Lie algebra, are the same as for the Volterra equation.

4. Contraction of the symmetry algebras in the continuous limit

It is well known [16, 17, 28] that the Toda equation has the potential Korteweg–de Vries equation as a continuous limit. The limit for the Volterra and higher Volterra equation is the Korteweg–de Vries equation itself. In the following we will consider each case separately.

4.1. The Toda equation

Let us consider the Toda equation (1.2). By setting

$$u_n(t) = -\tfrac{1}{2}hv(x,\tau) \tag{4.1}$$

$$x = (n-t)h \tag{4.2}$$

$$\tau = -\tfrac{1}{24}h^3 t \tag{4.3}$$

we can write equation (1.2) as

$$(v_\tau - v_{xxx} - 3v_x^2)_x = \mathcal{O}(h^2) \tag{4.4}$$

i.e., the once differentiated potential Korteweg–de Vries equation. Let us now rewrite the symmetry generators in the new coordinate system defined by (4.1)–(4.3) and develop them for small h in Taylor series. We have

$$\hat{X}_0^{TE} = \{-v_x(x,\tau)h - \tfrac{1}{24}v_\tau(x,\tau)h^3\}\partial_v \tag{4.5}$$

$$\hat{X}_1^{TE} = \{-2v_x(x,\tau)h - \tfrac{1}{3}v_\tau(x,\tau)h^3 + \mathcal{O}(h^5)\}\partial_v \tag{4.6}$$

$$\hat{X}_2^{TE} = \{-4v_x(x,\tau)h - \tfrac{7}{6}v_\tau(x,\tau)h^3 + \mathcal{O}(h^5)\}\partial_v \tag{4.7}$$

$$\hat{Y}_0^{TE} = \{2[v(x,\tau) + xv_x(x,\tau) + 3\tau v_\tau(x,\tau)] + \mathcal{O}(h)\}\partial_v \tag{4.8}$$

$$\hat{Z}_{-1}^{TE} = -\frac{2}{h}\partial_v \tag{4.9}$$

$$\hat{Z}_0^{TE} = \frac{48}{h^4}\tau\partial_v \tag{4.10}$$

$$\hat{Z}_1^{TE} = \left\{-\frac{96}{h^4}\tau + \frac{4}{h^2}[x + 6\tau v_x(x,\tau)] + \mathcal{O}(1)\right\}\partial_v. \tag{4.11}$$

To obtain equations (4.6)–(4.8) we impose the condition that v satisfies the potential Korteweg–de Vries equation

$$v_\tau = v_{xxx} + 3v_x^2. \tag{4.12}$$

The point symmetry generators, written in the evolutionary form, for the potential Korteweg–de Vries equation (4.12) are

$$\hat{P}_0 = v_\tau(x,\tau)\partial_v \tag{4.13}$$

$$\hat{P}_1 = v_x(x,\tau)\partial_v \tag{4.14}$$

$$\hat{B} = [x + 6\tau v_x(x,\tau)]\partial_v \tag{4.15}$$

$$\hat{D} = [v(x,\tau) + xv_x(x,\tau) + 3\tau v_\tau(x,\tau)]\partial_v \tag{4.16}$$

$$\hat{\Gamma} = \partial_v \tag{4.17}$$

and their commutation table is

	$\hat{P}_0$	$\hat{P}_1$	$\hat{B}$	$\hat{D}$	$\hat{\Gamma}$
$\hat{P}_0$	0	0	$-6\hat{P}_1$	$-3\hat{P}_0$	0
$\hat{P}_1$		0	$-\hat{\Gamma}$	$-\hat{P}_1$	0
$\hat{B}$			0	$2\hat{B}$	0
$\hat{D}$				0	$-\hat{\Gamma}$
$\hat{\Gamma}$					0

(4.18)

We can write some symmetry generators for the Toda equation, obtained as linear combinations of the generators (4.5)–(4.11), such that in the continuous limit they go into the generators of point symmetries of the potential Korteweg–de Vries equation:

$$\tilde{P}_0 = \frac{4}{h^3}(2\hat{X}_0^{TE} - \hat{X}_1^{TE}) \tag{4.19}$$

$$\tilde{P}_1 = -\frac{1}{h}\hat{X}_0^{TE} \tag{4.20}$$

$$\tilde{B} = \frac{h^2}{4}(2\hat{Z}_0^{TE} + \hat{Z}_1^{TE}) \tag{4.21}$$

$$\tilde{D} = \tfrac{1}{2}\hat{Y}_0^{TE} \tag{4.22}$$

$$\tilde{\Gamma} = -\frac{h}{2}\hat{Z}_{-1}^{TE}. \tag{4.23}$$

Taking into account the commutation table between the generators $\hat{X}_0^{TE}$, $\hat{X}_1^{TE}$, $\hat{Z}_{-1}^{TE}$, $\hat{Z}_0^{TE}$, $\hat{Z}_1^{TE}$ and $\hat{Y}_0^{TE}$, given by (3.31), and the continuous limit of $\hat{X}_2^{TE}$, given by equation (4.7), we get:

	$\tilde{P}_0$	$\tilde{P}_1$	$\tilde{B}$	$\tilde{D}$	$\tilde{\Gamma}$
$\tilde{P}_0$	0	0	$-6\tilde{P}_1 + \mathcal{O}(h^2)$	$-3\tilde{P}_0 + \mathcal{O}(h^2)$	0
$\tilde{P}_1$		0	$-\tilde{\Gamma} + \mathcal{O}(h^2)$	$-\tilde{P}_1 + \mathcal{O}(h^2)$	0
$\tilde{B}$			0	$2\tilde{B} + \mathcal{O}(h^2)$	0
$\tilde{D}$				0	$-\tilde{\Gamma}$
$\tilde{\Gamma}$					0

(4.24)

To get the results contained in (4.24) we had to require that $\beta = -2$, $2\gamma + \delta = 0$, and $\omega = \sigma = 0$ in equations (3.31). Thus, we have reobtained, in the continuous limit, all point symmetries of the potential KdV equation. The limit partially fixes the previously undetermined constants in equations (3.31).

An important observation is that, to obtain all the point symmetries of the potential KdV equation, we need not only the point symmetries $\hat{X}_0^{TE}$, $\hat{Z}_0^{TE}$, $\hat{Z}_{-1}^{TE}$ and $\hat{Z}_1^{TE}$ of the Toda equation but also the higher symmetries $\hat{X}_1^{TE}$, $\hat{Y}_0^{TE}$.

The x-differentiated potential KdV equation (4.4) has a further set of point symmetries, namely $f(\tau)\partial_v$, where $f(\tau)$ is an arbitrary function of time. They simply reflect the fact that, if $v(x,\tau)$ is a solution, then so is $w(x,\tau) = v(x,\tau) + f(\tau)$. These symmetries are of no particular interest. To obtain them, we would have to consider limits of higher symmetries of the Toda equation (in the $\hat{Y}_k^{TE}$ series). Moreover, to obtain the symmetries (4.5)–(4.11) we have used equation (4.12), rather than the differentiated equation (4.4).

4.2. The Volterra and higher Volterra equations

Let us consider the Volterra equation (1.3). By setting

$$a_n(t) = 1 + h^2 q(x,\tau) \tag{4.25}$$

$$x = (n - 2t)h \tag{4.26}$$

$$\tau = -\tfrac{1}{3}h^3 t \tag{4.27}$$

we can write equation (1.3) as

$$q_\tau = q_{xxx} + 6qq_x + \mathcal{O}(h^2) \tag{4.28}$$

i.e., the Korteweg–de Vries equation. Let us now rewrite the symmetry generators in the new coordinate system defined by (4.25)–(4.27) and develop them in a Taylor series for small h. We have

$$\hat{X}_0^V = \{-2hq_x(x,\tau) - \tfrac{1}{3}h^3 q_\tau(x,\tau)\}\partial_q \tag{4.29}$$

$$\hat{X}_1^V = \{-8hq_x(x,\tau) - \tfrac{10}{3}h^3 q_\tau(x,\tau) + \mathcal{O}(h^5)\}\partial_q \tag{4.30}$$

$$\hat{Y}_0^V = \{2[2q(x,\tau) + xq_x(x,\tau) + 3\tau q_\tau(x,\tau)] + \mathcal{O}(h)\}\partial_q \tag{4.31}$$

$$\hat{Z}^V = \left\{\frac{1}{h^2}[1 + 6\tau q_x(x,\tau)] + \mathcal{O}(1)\right\}\partial_q. \tag{4.32}$$

The symmetry generators, written in the evolutionary form, for the Korteweg–de Vries equation (4.28) are

$$\hat{P}_0 = q_\tau(x,\tau)\partial_q \tag{4.33}$$

$$\hat{P}_1 = q_x(x,\tau)\partial_q \tag{4.34}$$

$$\hat{B} = [1 + 6\tau q_x(x,\tau)]\partial_q \tag{4.35}$$

$$\hat{D} = [2q(x,\tau) + xq_x(x,\tau) + 3\tau q_\tau(x,\tau)]\partial_q \tag{4.36}$$

and their commutation table is

	$\hat{P}_0$	$\hat{P}_1$	$\hat{B}$	$\hat{D}$
$\hat{P}_0$	0	0	$-6\hat{P}_1$	$-3\hat{P}_0$
$\hat{P}_1$		0	0	$-\hat{P}_1$
$\hat{B}$			0	$2\hat{B}$
$\hat{D}$				0

(4.37)

We can write down some new symmetry generators for the Volterra equation, obtained as linear combinations of the generators (4.29)–(4.32), such that in the continuous limit they go over to the point symmetry generators of the Korteweg–de Vries equation:

$$\tilde{P}_0 = \frac{1}{2h^3}(4\hat{X}_0^V - \hat{X}_1^V) \tag{4.38}$$

$$\tilde{P}_1 = \frac{1}{12h}(\hat{X}_1^V - 10\hat{X}_0^V) \tag{4.39}$$

$$\tilde{D} = \tfrac{1}{2}\hat{Y}_0^V \tag{4.40}$$

$$\tilde{B} = h^2\hat{Z}^V. \tag{4.41}$$

Taking into account the commutation table between the generators $\hat{X}_0^V$, $\hat{X}_1^V$, $\hat{Z}^V$ and $\hat{Y}_0^V$, (3.23) and (3.33) and the fact that the continuous limit of $\hat{X}_2^V$ is given by

$$\hat{X}_2^V = [-32hq_x(x,\tau) - \tfrac{64}{3}h^3 q_\tau(x,\tau) + \mathcal{O}(h^5)]\partial_q \tag{4.42}$$

we get

$$\begin{array}{c|cccc} & \tilde{P}_0 & \tilde{P}_1 & \tilde{B} & \tilde{D} \\ \hline \tilde{P}_0 & 0 & 0 & -6\tilde{P}_1 + \mathcal{O}(h^2) & -3\tilde{P}_0 + \mathcal{O}(h^2) \\ \tilde{P}_1 & & 0 & \mathcal{O}(h^4) & -\tilde{P}_1 + \mathcal{O}(h^2) \\ \tilde{B} & & & 0 & 2\tilde{B} + \mathcal{O}(h^2) \\ \tilde{D} & & & & 0 \end{array} \tag{4.43}$$

Comparing the commutation tables (4.15) and (4.21) we see that the infinite-dimensional Lie algebra generated by $\hat{X}_0^V$, $\hat{X}_1^V$, $\hat{Z}^V$ and $\hat{Y}_0^V$, reduces, in the continuous limit, when h goes to 0, to the Lie algebra of the point symmetries of the Korteweg–de Vries equation.

In the case of the higher Volterra equation (1.4), the Korteweg–de Vries equation (4.28) is obtained by setting

$$a_n(t) = 1 + h^2 q(x, \tau) \tag{4.44}$$

$$x = nh \tag{4.45}$$

$$\tau = -2h^3 t. \tag{4.46}$$

Let us now rewrite the symmetry generators in the new coordinate system, defined by equations (4.44)–(4.46), and develop them for small h in a Taylor series. We get the same representation in terms of the symmetry generators of the Korteweg–de Vries equation as for the Volterra equation. So the contraction table is the same.

5. Conclusions

There are several general conclusions to be drawn from this study. The main one is that, if we wish to study symmetries of difference equations on a fixed and untransformable lattice and wish to obtain all symmetries that exist in the continuous limit, then generalized symmetries must be considered together with point ones. For the Toda system, the Toda equation and the two Volterra equations studied above, we always observed the same patterns. Namely, the linear tools of integrability theory provide us with the infinite-dimensional Lie algebra of symmetries of the discrete equations considered. This algebra includes a very small subalgebra of point transformations. In the continuous limit, when we take the spacing parameter h to 0, the structure of the Lie algebra changes. After the contraction $h \to 0$ it is still infinite-dimensional and still includes both generalized and point symmetries. However, a set of elements of the symmetry algebra of the discrete equation, that include point and generalized symmetries, contracts into purely point symmetries of the corresponding differential equation. For the Toda equation this is demonstrated in equations (4.19)–(4.24), and for the Volterra equation in equations (4.38)–(4.43).

We remark that, while Lie algebra contractions have many applications in physics [4], to our knowledge this is the first case when contractions of infinite-dimensional Lie algebras occur.

Another general result demonstrated in this paper is that for integrable equations, be they discrete, or continuous, it is very advantageous to analyse symmetries in the space of spectral data, and then to transfer the results into the usual phase space. This was done in section 3 when we calculated commutation relations for symmetries of the Toda and Volterra hierarchies.

A question that immediately arises is: how does one find and use symmetries of nonintegrable difference equations, when no linear system is available? One possibility is to give up the notion of a fixed lattice, i.e. study group transformations acting simultaneously on the equations and on the lattice. For instance, dilations of coordinates will change the scale of the lattice. This approach has been taken by Dorodnitsyn and collaborators [6, 7].

Another approach is the *differential equation method* proposed earlier [19]. Here one views a differential–difference equation as an infinite set of differential equations and looks for point symmetries of this infinite system. These will include some, through not necessarily all, generalized symmetries of the differential–difference equation under study. The problem here is that it is often very difficult to solve the corresponding infinite set of determining equations.

Finally, we mention the *intrinsic method* [19] which is the simplest to apply and use, and which provides us with the point symmetries of the discrete equation (on a fixed lattice).

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